Central limit theorems for a supercritical branching process in a random environment

Hesong Wang, Zhiqiang Gao, Quansheng Liu

Central limit theorems for a supercritical branching process in a random environment

Hesong WANG[a,b], Zhiqiang GAO[c,d,e], Quansheng LIU[b,d,e]

[a] *College of Mathematics and Computer Science, Hunan Normal University, Changsha, 410076 Hunan, China*
[b] *College of Mathematics and Computing Science, Changsha University of Science and Technology, Changsha, 410076 Hunan, China*
[c] *School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, 100875 Beijing, China*
[d] *LMAM, Université de Bretagne Sud, Campus de Tohannic, BP 573, 56017 Vannes, France*
[e] *Université Européenne de Bretagne, France*

Abstract

For a supercritical branching process (Z_n) in a stationary and ergodic environment ξ, we study the rate of convergence of the normalized population $W_n = Z_n/E[Z_n|\xi]$ to its limit W_∞: we show a central limit theorem for $W_\infty - W_n$ with suitable normalization and derive a Berry-Esseen bound for the rate of convergence in the central limit theorem when the environment is independent and identically distributed. Similar results are also shown for $W_{n+k} - W_n$ for each fixed $k \in \mathbb{N}^*$.

Keywords: Branching processes, random environment, central limit theorem, martingale, rate of convergence.
2000 MSC: 60J80, 60F05

1. Introduction

Galton-Watson processes have been studied by many authors, due to a wide range of applications. See for example the books by Harris (1963) and Athreya and Ney (1972). In a Galton-Watson process $\{Z_n, n = 0, 1, ...\}$, particles behave independently, each gives birth to a random number of particles of the next generation with a fixed distribution $\{p_k : k = 0, 1, ...\}$.

A branching process in a random environment is a natural and important extension of the Galton-Watson process. It is a class of non-homogeneous Galton-Watson processes indexed by a time-environment $\xi = (\xi_0, \xi_1, \xi_2, ...)$, which is

Email addresses: hesongwang@tom.com (Hesong WANG), gaozq@bnu.edu.cn (Zhiqiang GAO), quansheng.liu@univ-ubs.fr (Quansheng LIU)
[1] *Corresponding author:* GAO Zhiqiang, School of Mathematical Sciences, Beijing Normal University, 100875 Beijing, China.

supposed to be stationary and ergodic; given the environment ξ, the particles of n-th generation have offspring distribution $\{p_k(\xi_n) : k \in \mathbb{N}\}$ depending on ξ_n. For first important works on the subject, see Smith and Wilkinson (1969) and Athreya and Karlin (1971a,b).

For a Galton-Waston process with $Z_0 = 1$ and $m = EZ_1 \in (0, \infty)$, it is well known that $\{W_n = Z_n/m^n : n = 0, 1, ...\}$ forms a non-negative martingale, and converges almost surely to a random variable W_∞. For the convergence rate of the martingale, Heyde (1971) and Bühler (1969) obtained respectively that if $\mathrm{Var}(Z_1) = \sigma^2 < \infty$, then conditioned on $Z_n > 0$, the conditional laws of

$$(m^2 - m)^{\frac{1}{2}} \sigma^{-1} Z_n{}^{-\frac{1}{2}} m^n (W_\infty - W_n)$$

and

$$\big(m^k/(m^k - 1)\big)^{\frac{1}{2}} (m^2 - m)^{\frac{1}{2}} \sigma^{-1} Z_n{}^{-\frac{1}{2}} m^n (W_{n+k} - W_n) \qquad k \in \mathbb{N}^*$$

converge to the normal law $\mathcal{N}(0,1)$; Heyde and Brown (1971) gave an estimation of its convergence rate under a third moment condition.

The object of this paper is to extend the theorems of Bühler (1969), Heyde (1971) and Heyde and Brown (1971) to a branching process in a random environment. The main results are Theorems 2.1 and 2.2.

2. Main Results

As usual, we write $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{N}^* = \{1, 2, \cdots\}$ and $\mathbb{R}$ for the set of real numbers.

Let us first recall the definition of a branching process in a random environment. For reference on the subject, see for example Athreya and Karlin (1971a,b), and Athreya and Ney (1972).

A random environment $\xi = (\xi_n)$ is formulated as a stationary and ergodic sequence of random variables taking values in some measurable space $(\Theta, \mathcal{F})$. Each realization of ξ_n corresponds to a probability distribution $p(\xi_n) = \{p_i(\xi_n) : i \in \mathbb{N}\}$ where

$$p_i(\xi_n) \geq 0, \quad \sum_{i=0}^{\infty} p_i(\xi_n) = 1, \quad 0 < \sum_{i=0}^{\infty} i p_i(\xi_n) < \infty. \tag{1}$$

Without loss of generality, we can take ξ_n as coordinate functions defined on the product space $(\Theta^{\mathbb{N}}, \mathcal{F}^{\otimes\mathbb{N}})$, equipped with a probability law τ, which is invariant and ergodic under the usual shift transformation θ on $\Theta^{\mathbb{N}}$: $\theta(\xi_0, \xi_1, \cdots) = (\xi_1, \xi_2, \cdots)$. A branching process $(Z_n)_{n\geq 0}$ in the random environment ξ is a class of non-homogeneous branching processes indexed by ξ. By definition,

$$Z_0 = 1, \qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \;\; n \geq 0, \tag{2}$$

where given ξ, $\{X_{n,i} : n \geq 0, i \geq 1\}$ is a family of (conditionally) independent random variables, each $X_{n,i}$ has the common law $p(\xi_n)$. Notice that when all ξ_n are the same constant, (Z_n) reduces to the classical Galton-Waston process.

Let (Γ, P_ξ) be the probability space under which the process is defined when the environment ξ is given. As usual, P_ξ is called *quenched law*. The total probability space can be formulated as the product space $(\Gamma \times \Theta^{\mathbb{N}}, P)$, where $P = P_\xi \otimes \tau$ in the sense that for all measurable and positive function g, we have

$$\int g dP = \int \int g(\xi, y) dP_\xi(y) d\tau(\xi), \tag{3}$$

(recall that τ is the law of the environment ξ). The total probability P is usually called *annealed law*. The quenched law P_ξ may be considered to be the conditional probability of the annealed law P given ξ. The expectation with respect to P_ξ (resp. P) will be denoted E_ξ (resp. E).

For $n \geq 0$, define

$$m_n(a) = m(\xi_n, a) = \sum_{i=1}^{\infty} i^a p_i(\xi_n), \quad a \in \mathbb{R}, \tag{4}$$

$$m_n = m_n(1), \quad \sigma_n^2 = m_n(2) - m_n^2, \tag{5}$$

$$\pi_0 = 1 \quad \textit{and} \quad \pi_n = \pi_n(\xi) = m_0 \cdots m_{n-1} \quad \text{for} \quad n \geq 1. \tag{6}$$

Then $\pi_n = E_\xi Z_n$ for $n \geq 0$. It is well known that

$$W_n = Z_n / \pi_n \tag{7}$$

is a martingale with respect to the filtration

$$\mathcal{F}_0 = \{\varnothing, \Omega\}, \quad \mathcal{F}_n = \sigma\{\xi, X_{j,i} : j \leq n-1, i \geq 1\} \ \ (n \geq 1), \tag{8}$$

so that the limit

$$W_\infty = \lim_{n \to \infty} W_n \tag{9}$$

exists almost surely (a.s.) with $EW \leq 1$ by Fatou's lemma.

Throughout the paper, we always assume that

$$E \ln m_0 > 0 \quad \text{and} \quad E\left(\frac{Z_1}{m_0} \ln^+ Z_1\right) < \infty. \tag{10}$$

The first assumption ensures that the process is *supercritical* (cf. Athreya and Karlin (1971a)); the second one together with the first implies that $EW_\infty = 1$; moreover,

$$P_\xi(W_\infty > 0) = P_\xi(Z_n \to \infty) = \lim_{n \to \infty} P_\xi(Z_n > 0) = 1 - q(\xi) > 0 \quad \text{a.s.}, \tag{11}$$

where $q(\xi) = \lim_{n \to \infty} P_\xi(Z_n = 0)$ is the extinct probability.

In this paper, we search for central limit theorems on $W_\infty - W_n$ and $W_{n+k} - W_n$ for fixed $k \geq 1$ with an appropriate normalization. Assum that $m_0(2) < \infty$ a.s., and let

$$\Delta_k^2 = \Delta_k^2(\xi) = \sum_{0 \leq i < k} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2} \qquad \text{for} \quad k \in \mathbb{N}^* \cup \{\infty\}.$$

Then for $k \in \mathbb{N}^*$, $\Delta_k^2(\xi)$ is the variance of W_k under P_ξ; $\Delta_\infty^2(\xi)$ is the variance of W_∞ if the series converges (i.e. $\Delta_\infty^2(\xi) < \infty$): see Lemma 3.2.

We can now formulate our first main result.

Theorem 2.1. *Suppose that (10) holds and that $m_0(2) < \infty$ a.s.. In the case where $k = \infty$, assume additionally that $E \ln^+(\sigma_0^2/m_0^2) < \infty$. Write*

$$U_{n,k} = \frac{\pi_n (W_{n+k} - W_n)}{\sqrt{Z_n} \Delta_k(\theta^n \xi)} \qquad for \quad k \in \mathbb{N}^* \cup \{\infty\},$$

where by convention $W_{n+k} = W_\infty$ if $k = \infty$. Then for each $k \in \mathbb{N}^* \cup \{\infty\}$, as $n \to \infty$,

$$\sup_{x \in \mathbb{R}} |P_\xi(U_{n,k} \leq x | Z_n > 0) - \Phi(x)| \to 0 \quad in \quad L^1, \tag{12}$$

and

$$\sup_{x \in \mathbb{R}} |P(U_{n,k} \leq x | Z_n > 0) - \Phi(x)| \to 0. \tag{13}$$

We believe that for each $k \in \mathbb{N} \cup \{\infty\}$

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |P_\xi(U_{n,k} \leq x | Z_n > 0) - \Phi(x)| = 0 \quad \text{a.s..}$$

We notice that in the classical Galton-Waston process, (13) reduces to the results of Bühler (1969) and Heyde (1971). Our second main result concerns the rate of convergence in the above central limit theorem for a branching process with an independent and identically distributed environment.

Theorem 2.2. *Let the environment $\{\xi_n\}$ be independent and identically distributed. Assume that (10) holds and that $m_0(2) < \infty$ a.s.. In the case where $k = \infty$, assume additionally that $E \ln^+(\sigma_0^2/m_0^2) < \infty$. For each $k \in \mathbb{N}^* \cup \{\infty\}$, if $E|\frac{W_k - 1}{\Delta_k}|^{2+\delta} < \infty$ for some $\delta \in (0, 1]$, then*

$$\sup_{x \in \mathbb{R}} |P(U_{n,k} \leq x | Z_n > 0) - \Phi(x)| \leq \frac{C_\delta \left(E m_0(-\frac{\delta}{2})\right)^n E \left| \frac{W_k - 1}{\Delta_k} \right|^{2+\delta}}{P(Z_n > 0)}, \tag{14}$$

where $U_{n,k}$ is defined in Theorem 2.1 and C_δ is the Berry-Esseen constant.

Remark 2.3. It maybe useful to notice that if

$$E(Z_1/m_0)^{2+\delta} < \infty, \quad E{m_0}^{-(1+\delta)} < 1 \quad and \quad m_0(2)/m_0^2 \geq A$$

for some constant $A > 1$, then $E\left|\frac{W_\infty - 1}{\Delta_\infty}\right|^{2+\delta} < +\infty$. In fact by Theorem 3 of Guivarc'h and Liu (2001), the first two conditions imply that $E|W_\infty - 1|^{2+\delta} < \infty$, while the last one implies that $\Delta_\infty^2 \geq A - 1 > 0$.

For the classical Galton-Watson process with $\delta = 1$, Theorem 2.2 reduces to Theorem 2 of (Heyde and Brown, 1971, p.272).

3. Proof of Theorem 2.1

In this section, we consider a central limit theorem under a second moment condition in proving Theorem 2.1. We first give some lemmas.

Lemma 3.1 (Grincevičjus (1974))**.** *Let $\{(\alpha_n, \beta_n), n = 0, 1, 2, \cdots\}$ be a stationary and ergodic sequence of random variables with values in $\mathbb{R}^2$. If*

$$E \ln |\alpha_0| < 0 \quad \textit{and} \quad E \ln^+ |\beta_0| < \infty,$$

then

$$\sum_{n=0}^{\infty} |\alpha_0 \alpha_1 \cdots \alpha_{n-1} \beta_n| < \infty \quad a.s.$$

In fact, the result is a direct consequence of the ergodic theorem and Cauchy's criterion for the convergence of series.

Using the above lemma, we can easily obtain the following result.

Lemma 3.2. *Under the assumptions in Theorem 2.1, for each $k \in \mathbb{N}^* \cup \{\infty\}$,*

$$\mathrm{Var}_\xi(W_k) = \Delta_k^2(\xi) = \sum_{0 \leq i < k} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2}. \tag{15}$$

This has been known for branching processes in varying environment, see e.g. (Jagers, 1974, p.175) in a slightly different form. For reader's convenience, we present a proof in the following.

Proof of Lemma 3.2. By (2) and the definition of W_n, we have

$$W_{n+1} - W_n = \frac{1}{\pi_n} \sum_{j=1}^{Z_n} \left(\frac{X_{n,j}}{m_n} - 1\right).$$

Recall that under P_ξ, the random variables $\{X_{n,j}\}$ are independent of each other and have the common distribution $p(\xi_n)$ with expectation m_n. Hence a direct calculation shows that

$$\begin{aligned} E_\xi((W_{n+1} - W_n)^2) &= E_\xi\left(E_\xi((W_{n+1} - W_n)^2 | \mathcal{F}_n)\right) \\ &= E_\xi\left(\frac{Z_n}{{\pi_n}^2} \frac{\sigma_n^2}{m_n^2}\right) = \frac{1}{\pi_n} \frac{\sigma_n^2}{m_n^2}. \end{aligned}$$

As $\{W_n\}$ is a martingale, it follows that

$$E_\xi W_k^2 = E_\xi W_0^2 + \sum_{i=0}^{k-1} E((W_{i+1} - W_i)^2) = 1 + \sum_{i=0}^{k-1} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2}.$$

Therefore for each fixed integer k,

$$\mathrm{Var}_\xi(W_k) = E_\xi(W_k^2) - 1 = \sum_{i=0}^{k-1} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2}.$$

Now we turn to the calculation of $\mathrm{Var}_\xi(W_\infty)$. By Lemma 3.1, when $E \ln m_0 > 0$ and $E \ln^+ \frac{\sigma_i^2}{m_i^2} < \infty$,

$$\sup_n E_\xi(W_n^2) = 1 + \sum_{i=0}^{\infty} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2} < \infty \quad \text{a.s.}$$

So W_n converges to W_∞ in L^2 under P_ξ and

$$E_\xi(W_\infty^2) = \lim_{k\to\infty} E_\xi(W_k^2) = 1 + \sum_{i=0}^{\infty} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2}.$$

It follows that

$$\mathrm{Var}_\xi(W_\infty) = E_\xi(W_\infty^2) - 1 = \Delta_\infty^2(\xi) = \sum_{i=0}^{\infty} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2} < \infty \quad \text{a.s.}$$

□

To give our next lemma, we will need some notations, which will also be used in the proof of the main theorems. By definition,

$$Z_{n+k} = \sum_{j=1}^{Z_n} Z_k(n, j), \tag{16}$$

where $Z_k(n, j)$ denotes the number of descendants in the $(n + k)$-th generation of the j-th particle among the Z_n particles in n-th generation.

Writing $W_k(n, j) = \frac{Z_k(n,j)}{\pi_k(\theta^n \xi)}$ and using (16), we obtain the following decomposition:

$$\pi_n(W_{n+k} - W_n) = \sum_{j=1}^{Z_n} (W_k(n, j) - 1). \tag{17}$$

Letting $k \to \infty$, it follows that

$$\pi_n(W_\infty - W_n) = \sum_{j=1}^{Z_n} (W_\infty(n, j) - 1), \tag{18}$$

where under P_ξ, the random variables $\{W_\infty(n,j)\}_j$ are independent of each other and have the common conditional distribution

$$P_\xi(W_\infty(n,j) \in \cdot) = P_{\theta^n \xi}(W_\infty \in \cdot).$$

Lemma 3.3. *Suppose that the assumptions of Theorem 2.1 hold. Let $r_n \in \mathbb{N}$ with $r_n \to \infty$. For $k \in \mathbb{N}^* \cup \{\infty\}$, define*

$$Y_{k,n} = \frac{1}{\sqrt{r_n}} \sum_{j=1}^{r_n} \frac{W_k(n,j) - 1}{\Delta_k(\theta^n \xi)}.$$

Fix $k \in \mathbb{N}^* \cup \{\infty\}$. Then for each subsequence $\{n'\}$ of $\mathbb{N}$ with $n' \to \infty$, there is a subsequence $\{n''\}$ of $\{n'\}$ with $n'' \to \infty$ such that for a.e. ξ and all $x \in \mathbb{R}$, as $n'' \to \infty$,

$$P_\xi(Y_{k,n''} \le x) \to \Phi(x).$$

Proof. Fix $k \in \mathbb{N}^* \cup \{\infty\}$. In order to use Lindeberg's theorem, for $n \in \mathbb{N}$ and $\epsilon > 0$, we consider the quantity

$$L_k(\xi, \epsilon, n) = \frac{1}{r_n} \sum_{j=1}^{r_n} E_\xi \left(\left(\frac{W_k(n,j) - 1}{\Delta_k(\theta^n \xi)} \right)^2 ; \left| \frac{W_k(n,j) - 1}{\Delta_k(\theta^n \xi) \sqrt{r_n}} \right| > \epsilon \right),$$

where for a set A, we write $E_\xi(x; A)$ for $E_\xi(X \mathbf{1}_A)$, $\mathbf{1}_A$ denoting the indicator function of A. By the stationarity and ergodicity of the environment, for all $\epsilon > 0$, as $n \to \infty$,

$$EL_k(\xi, \epsilon, n) = E\left[\left(\frac{W_k - 1}{\Delta_k} \right)^2 ; \left| \frac{W_k - 1}{\Delta_k} \right| > \sqrt{r_n} \epsilon \right] \to 0. \tag{19}$$

Let $\{n'\}$ be a subsequence of $\mathbb{N}$. Notice that from (19), we can choose a subsequence $\{n''\}$ for which $L_k(\xi, \varepsilon, n'') \to 0$ a.s., but this sequence may depend of ϵ. We will use a diagonal argument to select a subsequence $\{n''\}$ of $\{n'\}$ such that a.s. $L_k(\xi, \varepsilon, n'') \xrightarrow{n'' \to \infty} 0$ *for all* $\epsilon > 0$. Set

$$\epsilon_m = 1/m \quad \text{for} \quad m \ge 1.$$

Let $\{n_{0,i}\} = \{n'\}$. Because of (19), there is a subsequence $\{n_{1,i}\}$ of $\{n_{0,i}\}$ and a set Λ_1 with $\tau(\Lambda_1) = 1$ such that $\forall \xi \in \Lambda_1$,

$$\lim_{i \to \infty} L_k(\xi, \epsilon_1, n_{1,i}) = 0.$$

Inductively for $m \ge 1$, when Λ_m and $\{n_{m,i}\}$ are defined such that $\tau(\Lambda_m) = 1$ and $\forall \xi \in \Lambda_m$, $L_k(\xi, \epsilon_m, n_{m,i}) \to 0$, there is a subsequence $\{n_{m+1,i}\} \subset \{n_{m,i}\}$ and a set Λ_{m+1} with $\tau(\Lambda_{m+1}) = 1$ such that $\forall \xi \in \Lambda_{m+1}$,

$$\lim_{i \to \infty} L_k(\xi, \epsilon_{m+1}, n_{m+1,i}) = 0.$$

We now consider the diagonal sequence $\{n_{i.i}\}_{i\geq 1}$ and $\Lambda = \bigcap_{j=1}^{\infty} \Lambda_j$. For each fixed $\epsilon > 0$, let $m \geq \frac{1}{\epsilon}$. Then $\epsilon_m \leq \epsilon$ and by the monoticity of $L_k(\xi, \epsilon, n)$ in ϵ, we see that $\forall \xi \in \Lambda$,

$$L_k(\xi, \epsilon, n_{m,i}) \leq L_k(\xi, \epsilon_m, n_{m,i}) \to 0 \quad \text{as} \quad i \to \infty.$$

As $\{n_{i,i}\}$ is a subsequence of $\{n_{m,i}\}$ whenever $i > m$, this implies that

$$\lim_{i\to\infty} L_k(\xi, \epsilon, n_{i,i}) = 0. \tag{20}$$

Since $\tau(\Lambda) = 1$, we have shown that for all $\epsilon > 0$, (20) holds a.s.. It follows that a.s. (20) holds for all rational $\epsilon > 0$, and therefore for all real $\epsilon > 0$ by the monoticity of $L_k(\xi, \epsilon, n_{ii})$ in ϵ. So by Lindeberg's theorem, it is a.s. that for all $x \in \mathbb{R}$, as $i \to \infty$,

$$P_\xi(Y_{k,n_{i,i}} \leq x) \to \Phi(x).$$

Thus the lemma has been proved with $\{n''\} = \{n_{i,i}\}$. □

Proof of Theorem 2.1. We shall only deal with the case where $k = \infty$, as the case where $k \in \mathbb{N}^*$ can be treated similarly.

We first prove the following assertion: *for each sequence $\{n'\}$ of $\mathbb{N}$ with $n' \to \infty$, there exist a subsequence $\{n''\}$ of $\{n'\}$ with $n'' \to \infty$ such that for a.e. ξ and all x, as $n'' \to \infty$,*

$$P_\xi(U_{n'',\infty} \leq x | Z_{n''} > 0) \to \Phi(x). \tag{21}$$

By the definition of $U_{n,\infty}$ and the relation (18), we get :

$$U_{n,\infty} = \frac{\pi_n(W_\infty - W_n)}{\sqrt{Z_n}\Delta_\infty(\theta^n\xi)} = \frac{1}{\sqrt{Z_n}} \sum_{j=1}^{Z_n} \frac{W_\infty(n,j) - 1}{\Delta_\infty(\theta^n\xi)},$$

where we recall that under P_ξ, $\{W_\infty(n,j), j \geq 1\}$ is a family of random variables independent of each other and independent of Z_n, each has the same law as W_∞ under $P_{\theta^n\xi}$. Set

$$u_n(r,x) = P_\xi\left(\frac{1}{\sqrt{r}} \sum_{j=1}^{r} \frac{W_\infty(n,j) - 1}{\Delta_\infty(\theta^n\xi)} \leq x\right), \quad r \in \mathbb{N}^*, \quad x \in \mathbb{R}.$$

Then

$$\begin{aligned} P_\xi(U_{n,\infty} \leq x | Z_n > 0) &= [P_\xi(Z_n > 0)]^{-1} \sum_{r=1}^{\infty} P_\xi(U_{n,\infty} \leq x, Z_n = r) \\ &= \sum_{r=1}^{\infty} u_n(r,x) \frac{P_\xi(Z_n = r)}{P_\xi(Z_n > 0)}. \end{aligned} \tag{22}$$

To show the main idea, let us first consider the special case where $q(\xi) = 0$ a.s., i.e. for a.e. ξ,

$$Z_n \to \infty \qquad P_\xi^*\text{-a.s..}$$

In this case, the relation (22) becomes

$$P_\xi(U_{n,\infty} \le x) = \sum_{r=1}^{\infty} u_n(r,x) P_\xi(Z_n = r) = E_\xi u_n(Z_n, x).$$

By Lemma 3.3, for each subsequence $\{n'\}$ of $\mathbb{N}$ with $n' \to \infty$, there exist a subsequence $\{n''\}$ of $\{n'\}$ with $n'' \to \infty$ such that for a.e. ξ and all x, as $n'' \to \infty$,

$$u_{n''}(Z_{n''}, x) \to \Phi(x).$$

By the dominated convergence theorem, for a.e. ξ and all x, as $n'' \to \infty$,

$$P_\xi(U_{n'',\infty} \le x) = E_\xi[u_{n''}(Z_{n''}, x)] \to \Phi(x).$$

So we have proved (21) when $q(\xi) = 0$ a.s..

We now consider the general case where $0 \le q(\xi) < 1$ a.s..

For each $\xi \in \Theta^{\mathbb{N}}$, let Z_n^* be random variables defined on some probability space $(\Gamma^*, \mathbb{P}_\xi^*)$ with law

$$P_\xi^*(Z_n^* = r) = \frac{P_\xi(Z_n = r)}{P_\xi(Z_n > 0)}, \qquad r \in \mathbb{N}^*.$$

Then

$$P_\xi(U_{n,\infty} \le x | Z_n > 0) = E_\xi^* u_n(Z_n^*, x),$$

where E_ξ^* denotes the expectation with respect to P_ξ^*.

Let $\{n'\}$ be a sequence of $\mathbb{N}$ with $n' \to \infty$. If for a.e. ξ,

$$Z_{n'}^* \to \infty \qquad P_\xi^*\text{-a.s.,}$$

then as above we can use Lemma 3.3 and the dominated convergence theorem to show that there is a sequence $\{n''\}$ of $\{n'\}$ with $n'' \to \infty$ such that for all x, as $n'' \to \infty$,

$$E_\xi^* u_{n''}(Z_{n''}^*, x) \to \Phi(x).$$

By the fact that $Z_n^* \to \infty$ in probability under P_ξ^*, we can choose a subsequence for which $Z_n^* \to \infty$ P_ξ^*-a.s.. But to apply Lemma 3.3, we need that the sequence does not depend on ξ. We therefore pass to the probability P^* to overcome this difficulty, where $P^* = P_\xi^* \otimes \tau$ is defined on the product space $\Gamma^* \times \Theta^{\mathbb{N}}$ just as P was defined on $\Gamma \times \Theta^{\mathbb{N}}$.

Notice that for each $r \in \mathbb{N}^*$, as $n \to \infty$,

$$P_\xi^*(Z_n^* = r) = \frac{P_\xi(Z_n = r)}{P_\xi(Z_n > 0)} \to 0,$$

where the last step holds as $Z_n \to \infty$ a.s. on the survival event $S = \{Z_n > 0, \forall n \geq 1\}$ (see (11) or Tanny (1977) for this fact). Then $Z_n^* \to +\infty$ in probability under P_ξ^*. By the dominated convergence theorem, this implies that $Z_n^* \to +\infty$ in probability under P^*. Therefore for each subsequence $\{n'\}$ of $\mathbb{N}$ with $n' \to \infty$, there is a subsequence $\{\tilde{n}\} \subset \{n'\}$ with $\tilde{n} \to \infty$ such that $Z_{\tilde{n}}^* \to +\infty$ a.s. under P^*. This implies that for a.e. ξ, as $\tilde{n} \to \infty$,

$$Z_{\tilde{n}}^* \to +\infty \quad P_\xi^*\text{-a.s.}$$

Now by Lemma 3.3, there exists a subsequence $\{n''\}$ of $\{\tilde{n}\}$ such that for a.e. ξ and all x, as $n'' \to \infty$,

$$u_{n''}(Z_{n''}^*, x) \to \Phi(x).$$

By the dominated convergence theorem, for almost every ξ and each x, as $n'' \to \infty$,

$$P_\xi(U_{n'',\infty} \leq x | Z_{n''} > 0) = E_\xi u_{n''}(Z_{n''}^*, x) \to \Phi(x). \tag{23}$$

So combining the above two cases, we have proved (21).

Since $P_\xi(U_{n'',\infty} \leq x | Z_{n''} > 0)$ are distribution functions and $\Phi(x)$ is a continuous distribution function, by Dini's Theorem we see that for a.e. ξ, as $n'' \to \infty$,

$$\sup_x |P_\xi(U_{n'',\infty} \leq x | Z_{n''} > 0) - \Phi(x)| \to 0. \tag{24}$$

By the dominated convergence theorem, (24) implies that as $n'' \to \infty$,

$$E \sup_x |P_\xi(U_{n'',\infty} \leq x | Z_{n''} > 0) - \Phi(x)| \to 0. \tag{25}$$

Therefore we have proved that for each sequence $\{n'\}$ of $\mathbb{N}$ with $n' \to \infty$, there is a subsequence $\{n''\}$ of $\{n'\}$ with $n'' \to \infty$ such that (25) holds. Hence

$$E \sup_x |P_\xi(U_{n,\infty} \leq x | Z_n > 0) - \Phi(x)| \to 0.$$

This gives (12) for $k = \infty$. The proof for $k \in \mathbb{N}^*$ is similar.

We now begin to prove (13).

As we have proved that for each subsequence $\{n'\}$ of $\mathbb{N}$, there is a subsequence $\{n''\}$ so that (24) holds, which implies: for a.e. ξ and all $x \in \mathbb{R}$, as $n'' \to \infty$,

$$|P_\xi(U_{n'',\infty} \leq x | Z_{n''} > 0) - \Phi(x)| \to 0.$$

It follows that for a.e. ξ and all $x \in \mathbb{R}$,

$$|P_\xi(U_{n'',\infty} \leq x, Z_{n''} > 0) - P_\xi(Z_{n''} > 0)\Phi(x)| \to 0.$$

So by the dominated convergence theorem, we see that for each $x \in \mathbb{R}$, as $n'' \to \infty$,

$$|P(U_{n'',\infty} \leq x, Z_{n''} > 0) - P(Z_{n''} > 0)\Phi(x)| \to 0,$$

and hence

$$P(U_{n'',\infty} \le x | Z_{n''} > 0) \to \Phi(x).$$

By Dini's Theorem, it follows that

$$\sup_x |P(U_{n'',\infty} \le x | Z_{n''} > 0) - \Phi(x)| \to 0. \tag{26}$$

Therefore we have proved that for each sequence $\{n'\}$ of $\mathbb{N}$, there is a subsequence $\{n''\}$ of $\{n'\}$ with $n'' \to \infty$ such that (26) holds. Hence

$$\sup_x |P(U_{n,\infty} \le x | Z_n > 0) - \Phi(x)| \to 0.$$

Thus the proof is completed. □

4. Proof of Theorem 2.2

In this section, we consider the rate of convergence in the central limit theorem under a moment condition of order $2 + \delta$, in proving Theorem 2.2.

Notice that by the definition (4) of $m_n(a)$, we have

$$m_n(a) = E_\xi X_{n,i}^a \quad \text{if} \quad a > 0, \quad m_n(a) = E_\xi X_{n,i}^a \mathbf{1}_{\{X_{n,i} > 0\}} \quad \text{if} \quad a \le 0, \tag{27}$$

where $X_{n,i}$ is as in (2). For $a > 0$, define

$$R_n = [m_0(-a) \cdots m_{n-1}(-a)]^{-1} Z_n^{-a} \mathbf{1}_{\{Z_n > 0\}}, \quad n \ge 0.$$

Lemma 4.1. *$(R_n, \mathcal{F}_n)_{n \ge 0}$ is a supermartingale, where $\mathcal{F}_n$ were defined in (8).*

Proof. Using the decomposition (2) of Z_{n+1}, we have

$$\begin{aligned} Z_{n+1}^{-a} \mathbf{1}_{\{Z_{n+1} > 0\}} &= \left[\sum_{i=1}^{Z_n} X_{n,i} \right]^{-a} \mathbf{1}_{\{Z_n > 0\}} \mathbf{1}_{\{Z_{n+1} > 0\}} \\ &= Z_n^{-a} \left[\frac{1}{Z_n} \sum_{i=1}^{Z_n} X_{n,i} \mathbf{1}_{\{X_{n,i}\} > 0} \right]^{-a} \mathbf{1}_{\{Z_n > 0\}} \mathbf{1}_{\{Z_{n+1} > 0\}} \\ &\le {Z_n}^{-a} \frac{1}{Z_n} \sum_{i=1}^{Z_n} (X_{n,i} \mathbf{1}_{\{X_{n,i}\} > 0})^{-a} \mathbf{1}_{\{Z_n > 0\}} \mathbf{1}_{\{Z_{n+1} > 0\}}, \end{aligned}$$

where the last inequality is due to the convexity property of the function $x^{-a} (a > 0)$.

Taking conditional expectation with respect to $\mathcal{F}_n$ and P_ξ on both sides of the above inequality, we obtain that

$$E_\xi (Z_{n+1}^{-a} \mathbf{1}_{\{Z_{n+1} > 0\}} | \mathcal{F}_n) \le Z_n^{-a} \mathbf{1}_{\{Z_n > 0\}} m_n(-a), \tag{28}$$

which gives the desired result. □

Since $Z_0 = 1$, by (28), we immediate obtain the following

Lemma 4.2. *For $a > 0$, we have*

$$E_\xi Z_n^{-a} \mathbf{1}_{\{Z_n>0\}} \leq m_0(-a) \cdots m_{n-1}(-a) \tag{29}$$

If the environment sequence $\{\xi_n\}$ is independent and identically distributed, then

$$EZ_n^{-a} \mathbf{1}_{\{Z_n>0\}} \leq (Em_0(-a))^n. \tag{30}$$

Now we give the proof of Theorem 2.2.

Proof of Theorem 2.2. We shall only deal with the case $k = \infty$, as the case where $k \in \mathbb{N}^*$ can be treated similarly.

Consider the probability space $(\Gamma^* \times \Theta^{\mathbb{N}}, P^*)$ and define random variables Z_n^* as in the proof of Theorem 2.1. By definition,

$$u_n(Z_n^*, x) = P_\xi \left(\frac{1}{\sqrt{Z_n^*}} \sum_{j=1}^{Z_n^*} \frac{W_\infty(n,j) - 1}{\Delta_\infty(\theta^n \xi)} \leq x \right).$$

By our hypothesis and the Berry-Esseen theorem (see e.g. Theorem 6 of (Petrov, 1995, p.115)), we have

$$\begin{aligned} |u_n(Z_n^*, x) - \Phi(x)| &\leq \frac{C_\delta}{(Z_n^*)^{1+\frac{\delta}{2}}} \sum_{j=1}^{Z_n^*} E_\xi \left| \frac{W_\infty(n,j) - 1}{\Delta_\infty(\theta^n \xi)} \right|^{2+\delta} \\ &= C_\delta (Z_n^*)^{-\frac{\delta}{2}} E_{\theta^n \xi} \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta}, \end{aligned}$$

where C_δ is the Berry-Esseen constant. Using this evaluation, we can derive that

$$\begin{aligned} |P_\xi(U_{n,\infty} \leq x | Z_n > 0) - \Phi(x)| &\leq E_\xi^* |u_n(Z_n^*, x) - \Phi(x)| \\ &\leq C_\delta E_\xi^* (Z_n^*)^{-\frac{\delta}{2}} E_{\theta^n \xi} \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta}. \end{aligned}$$

By the definition of Z_n^*, this implies that

$$\begin{aligned} &|P_\xi(U_{n,\infty} \leq x, Z_n > 0) - P_\xi(Z_n > 0)\Phi(x)| \\ &\qquad \leq C_\delta E_\xi \left(Z_n^{-\frac{\delta}{2}} I_{\{Z_n>0\}} \right) E_{\theta^n \xi} \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta}. \end{aligned} \tag{31}$$

Using (31) and the fact that the sequence $\{\xi_n\}$ is independent and identically distributed, we get

$$\begin{aligned} &|P(U_{n,\infty} \leq x, Z_n > 0) - P(Z_n > 0)\Phi(x)| \\ &\leq E|P_\xi(U_{n,\infty} \leq x, Z_n > 0) - P_\xi(Z_n > 0)\Phi(x)| \\ &\leq C_\delta E \left(Z_n^{-\frac{\delta}{2}} I_{\{Z_n>0\}} \right) E \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta}. \end{aligned}$$

Together with (30), we obtain that

$$|P(U_{n,\infty} \leq x | Z_n > 0) - \Phi(x)| \leq \frac{C_\delta (Em_0(-\frac{\delta}{2}))^n E|\frac{W_\infty - 1}{\Delta_\infty}|^{2+\delta}}{P(Z_n > 0)}.$$

Then the proof is completed. □

Acknowledgements

The authors would like to thank the Editor and an anonymous referee for their comments and remarks. The research was the supported by the National Natural Science Foundation of China (Grant No. 10771021, 10871064 and 50907005), the Key Labor. of Comput. Stoch. Math., Univ. of Hunan (No. 09K026) and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, Ministry of Education of China (Grant No. [2008] 890).

References

Athreya, K. B., Karlin, S., 1971a. On branching processes with random environments. I. Extinction probabilities. Ann. Math. Statist. 42, 1499–1520.

Athreya, K. B., Karlin, S., 1971b. Branching processes with random environments. II. Limit theorems. Ann. Math. Statist. 42, 1843–1858.

Athreya, K. B., Ney, P. E., 1972. Branching processes. Springer-Verlag, New York, die Grundlehren der mathematischen Wissenschaften, Band 196.

Bühler, W. J., 1969. Ein zentraler Grenzwertsatz für Verzweigungsprozesse. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 11, 139–141.

Grincevičjus, A. K., 1974. The continuity of the distribution of a certain sum of dependent variables that is connected with independent walks on lines. Teor. Verojatnost. i Primenen. 19, 163–168.

Guivarc'h, Y., Liu, Q., 2001. Propriétés asymptotiques des processus de branchement en environnement aléatoire. C. R. Acad. Sci. Paris Sér. I Math. 332 (4), 339–344.

Harris, T. E., 1963. The theory of branching processes. Die Grundlehren der Mathematischen Wissenschaften, Bd. 119. Springer-Verlag, Berlin.

Heyde, C. C., 1971. Some central limit analogues for supercritical Galton-Watson processes. J. Appl. Probability 8, 52–59.

Heyde, C. C., Brown, B. M., 1971. An invariance principle and some convergence rate results for branching processes. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 20, 271–278.

Jagers, P., 1974. Galton-Watson processes in varying environments. J. Appl. Probability 11, 174–178.

Petrov, V. V., 1995. Limit theorems of probability theory. Vol. 4 of Oxford Studies in Probability. The Clarendon Press Oxford University Press, New York, sequences of independent random variables, Oxford Science Publications.

Smith, W. L., Wilkinson, W. E., 1969. On branching processes in random environments. Ann. Math. Statist. 40, 814–827.

Tanny, D., 1977. Limit theorems for branching processes in a random environment. Ann. Probability 5 (1), 100–116.